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On one master integral for three-loop on-shell HQET propagator diagrams with mass

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ABSTRACT: An exact expression for the master integral I_2 [1] arising in three-loop on-shell HQET propagator diagrams with mass is derived and its analytical expansion in the dimensional regularization parameter ε is given.

KEYWORDS: NLO Computations.

All three-loop on-shell HQET propagator integrals with a loop of a massive quark can be reduced to a set of master integrals [1]. A reduction algorithm has been constructed by solving the integration-by-parts [2] relations using a Gröbner bases technique [3]. Some master integrals are known exactly, for the remaining ones a few terms of their respective ε expansion have been calculated in Ref. [1]. In particular, for the integral I_2, the $1/\varepsilon$ and $\mathcal{O}(\varepsilon^0)$ terms are known. However, in

Figure 1: The integral $I_{n_1 n_2 n_3}$

some applications more terms of its ε expansion are needed (A. Czarnecki, A. Pak, work in progress). Here, we shall obtain an exact result for I_2 and additional terms of its expansion in ε.
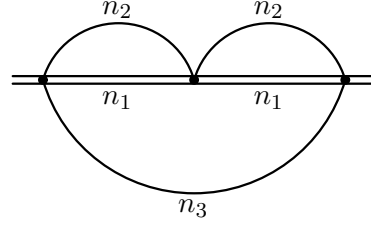
We shall consider a more general integral (Fig. 1):

$$
I_{n_1 n_2 n_3} = \frac{1}{i\pi^{d/2}} \int \frac{I_{n_1 n_2}^2(p_0)\, d^d p}{(1 - p^2 - i0)^{n_3}} \,,
$$

$$
I_{n_1 n_2}(p_0) = \frac{1}{i\pi^{d/2}} \int \frac{d^d k}{(-2(k_0 + p_0) - i0)^{n_1}(1 - k^2 - i0)^{n_2}} \,,
\tag{1}
$$

using a method similar to the one used in [4] for calculating a three-loop vacuum integral. Using the HQET Feynman parametrization, we write the one-loop subdiagram as

$$
I_{n_1 n_2}(p_0) = \frac{\Gamma(n_1 + n_2 - 2 + \varepsilon)}{\Gamma(n_1)\Gamma(n_2)} \int_0^\infty y^{n_1 - 1}(y^2 - 2p_0 y + 1)^{2 - n_1 - n_2 - \varepsilon}\, dy \,.
\tag{2}
$$

Then, after the Wick rotation $p_0 = ip_{E0}$, we can calculate the integral in $d^{d-1}\vec{p}$:

$$
I_{n_1 n_2 n_3} = \frac{\Gamma(n_3 - 3/2 + \varepsilon)}{\pi^{1/2}\Gamma(n_3)} \int_{-\infty}^{+\infty} I_{n_1 n_2}^2(ip_{E0})(1 + p_{E0}^2)^{3/2 - n_3 - \varepsilon}\, dp_{E0} \,.
\tag{3}
$$

Here $\operatorname{Re} I_{n_1 n_2}^2(ip_{E0})$ is an even function of p_{E0}; $\operatorname{Im} I_{n_1 n_2}^2(ip_{E0})$ is an odd function, and does not contribute to the integral.

The integral (2) at $p_0 < 0$ can be expressed via the hypergeometric function $_2F_1$:

$$
I_{n_1 n_2}(p_0) = \frac{\Gamma(n_1 + n_2 - 2 + \varepsilon)\Gamma(n_1 + 2n_2 - 4 + 2\varepsilon)}{\Gamma(n_2)\Gamma(2(n_1 + n_2 - 2 + \varepsilon))}{}_2F_1\left(\begin{array}{c} \frac{1}{2}n_1, \frac{1}{2}n_1 + n_2 - 2 + \varepsilon \\ n_1 + n_2 - \frac{3}{2} + \varepsilon \end{array} \middle| 1 - p_0^2 \right)
\tag{4}
$$

(a similar expression has been derived in Ref. [5]). Its square can be expressed via $_3F_2$, due to the Clausen identity. In order to perform an analytical continuation to $p_0 = ip_{E0}$, we re-express this $_3F_2$ via three $_3F_2$ functions of the inverse argument:

$$
\operatorname{Re} I_{n_1 n_2}^2(ip_{E0}) = R_{n_1 n_2}(z) \,, \qquad z = \frac{1}{1 + p_{E0}^2} \,,
\tag{5}
$$

where

$$
R_{n_1 n_2}(z) = \frac{\Gamma^2(n_2 - 2 + \varepsilon)}{\Gamma^2(n_2)}\left(-\frac{z}{4}\right)^{n_1}\left[{}_3F_2\left(\begin{array}{c} n_1, \frac{5}{2} - n_2 - \varepsilon, 5 - n_1 - 2n_2 - 2\varepsilon \\ 3 - n_2 - \varepsilon, 5 - 2n_2 - 2\varepsilon \end{array} \middle| z \right)\right.
$$

$$- 2A_{n_1 n_2}(z) \cos(\pi\varepsilon) \, {}_3F_2 \left(\begin{array}{c} \frac{1}{2}, 3 - n_1 - n_2 - \varepsilon, n_1 + n_2 - 2 + \varepsilon \\ 3 - n_2 - \varepsilon, n_2 - 1 + \varepsilon \end{array} \middle| z \right)$$

$$+ A_{n_1 n_2}^2(z) \cos(2\pi\varepsilon) \, {}_3F_2 \left(\begin{array}{c} 1 - n_1, n_2 - \frac{3}{2} + \varepsilon, n_1 + 2n_2 - 4 + 2\varepsilon \\ n_2 - 1 + \varepsilon, 2n_2 - 3 + 2\varepsilon \end{array} \middle| z \right) \Bigg], \quad (6)$$

and

$$A_{n_1 n_2}(z) = \frac{\pi}{\sin(\pi\varepsilon)} \frac{\Gamma(n_1 + 2n_2 - 4 + 2\varepsilon)}{\Gamma(n_1) h(n_2 - 2 + \varepsilon)} \left(\frac{z}{4} \right)^{n_2 - 2 + \varepsilon} \quad (7)$$

with $h(x) = x\,\Gamma^2(x)$. This result simplifies at $n_1 = 1$:

$$R_{1 n_2}(z) = -\frac{\Gamma^2(n_2 - 2 + \varepsilon)}{4\Gamma^2(n_2)} z \Bigg[{}_3F_2 \left(\begin{array}{c} 1, \frac{5}{2} - n_2 - \varepsilon, 4 - 2n_2 - 2\varepsilon \\ 3 - n_2 - \varepsilon, 5 - 2n_2 - 2\varepsilon \end{array} \middle| z \right)$$

$$- 2A_{1 n_2}(z) \cos(\pi\varepsilon) \, {}_2F_1 \left(\begin{array}{c} \frac{1}{2}, 2 - n_2 - \varepsilon \\ 3 - n_2 - \varepsilon \end{array} \middle| z \right) + A_{1 n_2}^2(z) \cos(2\pi\varepsilon) \Bigg]. \quad (8)$$

Calculating the integral

$$I_{n_1 n_2 n_3} = \frac{\Gamma(n_3 - 3/2 + \varepsilon)}{\pi^{1/2} \Gamma(n_3)} \int_0^1 R_{n_1 n_2}(z) z^{n_3 - 3 + \varepsilon} (1 - z)^{-1/2} dz \quad (9)$$

term-by-term, we obtain

$$I_{n_1 n_2 n_3} = \frac{(-1)^{n_1}}{\Gamma^2(n_2)\Gamma(n_3)} \Bigg[\frac{h(n_1 + n_3 - 2 + \varepsilon)\,\Gamma^2(n_2 - 2 + \varepsilon)\Gamma(2n_3 - 3 + 2\varepsilon)}{\Gamma(n_3 - 1 + \varepsilon)\Gamma(2n_1 + 2n_3 - 3 + 2\varepsilon)}$$

$$\times {}_4F_3 \left(\begin{array}{c} n_1, \frac{5}{2} - n_2 - \varepsilon, n_1 + n_3 - 2 + \varepsilon, 5 - n_1 - 2n_2 - 2\varepsilon \\ n_1 + n_3 - \frac{3}{2} + \varepsilon, 5 - 2n_2 - 2\varepsilon, 3 - n_2 - \varepsilon \end{array} \middle| 1 \right)$$

$$- 2\pi \frac{\cos(\pi\varepsilon)}{\sin(\pi\varepsilon)} \frac{h(n_1 + n_2 + n_3 - 4 + 2\varepsilon)\,\Gamma(n_1 + 2n_2 - 4 + 2\varepsilon)\Gamma(2n_3 - 3 + 2\varepsilon)}{(n_2 - 2 + \varepsilon)\Gamma(n_1)\Gamma(n_3 - 1 + \varepsilon)\Gamma(2n_1 + 2n_2 + 2n_3 - 7 + 4\varepsilon)}$$

$$\times {}_4F_3 \left(\begin{array}{c} \frac{1}{2}, n_1 + n_2 + n_3 - 4 + 2\varepsilon, 3 - n_1 - n_2 - \varepsilon, n_1 + n_2 - 2 + \varepsilon \\ n_1 + n_2 + n_3 - \frac{7}{2} + 2\varepsilon, 3 - n_2 - \varepsilon, n_2 - 1 + \varepsilon \end{array} \middle| 1 \right)$$

$$+ \pi^2 \frac{\cos(2\pi\varepsilon)}{\sin^2(\pi\varepsilon)} \frac{h(n_1 + 2n_2 + n_3 - 6 + 3\varepsilon)\,\Gamma^2(n_1 + 2n_2 - 4 + 2\varepsilon)\Gamma(2n_3 - 3 + 2\varepsilon)}{\Gamma^2(n_1)\Gamma^2(n_2 - 1 + \varepsilon)\Gamma(n_3 - 1 + \varepsilon)\Gamma(2n_1 + 4n_2 + 2n_3 - 11 + 6\varepsilon)}$$

$$\times {}_4F_3 \left(\begin{array}{c} 1 - n_1, n_2 - \frac{3}{2} + \varepsilon, n_1 + 2n_2 - 4 + 2\varepsilon, n_1 + 2n_2 + n_3 - 6 + 3\varepsilon \\ n_2 - 1 + \varepsilon, 2n_2 - 3 + 2\varepsilon, n_1 + 2n_2 + n_3 - \frac{11}{2} + 3\varepsilon \end{array} \middle| 1 \right) \Bigg]. (10)$$

In particular,

$$I_{1 n_2 n_3} = -\frac{1}{\Gamma^2(n_2)\Gamma(n_3)} \Bigg[\frac{\Gamma^2(n_2 - 2 + \varepsilon)\Gamma(n_3 - 1 + \varepsilon)}{2(2n_3 - 3 + 2\varepsilon)}$$

$$\times {}_4F_3 \left(\begin{array}{c} 1, \frac{5}{2} - n_2 - \varepsilon, n_3 - 1 + \varepsilon, 4 - 2n_2 - 2\varepsilon \\ n_3 - \frac{1}{2} + \varepsilon, 5 - 2n_2 - 2\varepsilon, 3 - n_2 - \varepsilon \end{array} \middle| 1 \right)$$

$$- 2\pi \frac{\cos(\pi\varepsilon)}{\sin(\pi\varepsilon)} \frac{h(n_2 + n_3 - 3 + 2\varepsilon)\,\Gamma(2n_2 - 3 + 2\varepsilon)\Gamma(2n_3 - 3 + 2\varepsilon)}{(n_2 - 2 + \varepsilon)\Gamma(n_3 - 1 + \varepsilon)\Gamma(2n_2 + 2n_3 - 5 + 4\varepsilon)}$$

$$\times {}_3F_2 \left(\begin{matrix} \frac{1}{2}, n_2 + n_3 - 3 + 2\varepsilon, 2 - n_2 - \varepsilon \\ n_2 + n_3 - \frac{5}{2} + 2\varepsilon, 3 - n_2 - \varepsilon \end{matrix} \middle| 1 \right)$$

$$+ \pi^2 \frac{\cos(2\pi\varepsilon)}{\sin^2(\pi\varepsilon)} \frac{h(2n_2 + n_3 - 5 + 3\varepsilon)\, \Gamma^2(2n_2 - 3 + 2\varepsilon)\Gamma(2n_3 - 3 + 2\varepsilon)}{\Gamma^2(n_2 - 1 + \varepsilon)\Gamma(n_3 - 1 + \varepsilon)\Gamma(4n_2 + 2n_3 - 9 + 6\varepsilon)} \Bigg]. \tag{11}$$

For example, let us consider the convergent integral I_{122}. Using the reduction procedure of Ref. [1], we can relate it to the master integral $I_2 \equiv I_{111}$:

$$I_{122} = -\frac{(d-3)^2(d-4)(3d-8)(3d-10)}{8(3d-11)(3d-13)} I_2 \,. \tag{12}$$

From Eq. (11) we have

$$\frac{I_{122}}{\Gamma^3(1+\varepsilon)} = -\frac{1}{2\varepsilon^2} \Bigg[\frac{1}{1+2\varepsilon} {}_4F_3 \left(\begin{matrix} 1, \frac{1}{2} - \varepsilon, 1 + \varepsilon, -2\varepsilon \\ \frac{3}{2} + \varepsilon, 1 - \varepsilon, 1 - 2\varepsilon \end{matrix} \middle| 1 \right)$$

$$- \frac{2}{1+4\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma^3(1+2\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1+4\varepsilon)} {}_3F_2 \left(\begin{matrix} \frac{1}{2}, 1 + 2\varepsilon, -\varepsilon \\ \frac{3}{2} + 2\varepsilon, 1 - \varepsilon \end{matrix} \middle| 1 \right)$$

$$+ \frac{1}{1+6\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma^4(1+2\varepsilon)\Gamma(1-2\varepsilon)\Gamma^2(1+3\varepsilon)}{\Gamma^4(1+\varepsilon)\Gamma(1+4\varepsilon)\Gamma(1-4\varepsilon)\Gamma(1+6\varepsilon)} \Bigg]. \tag{13}$$

There are several methods to expand the hypergeometric functions in ε [6, 7, 8]. We first follow the method of Ref. [7]. First we express Γ functions in the hypergeometric series as exponents containing S-sums

$$S(N; k_1, \ldots, k_l; x_1, \ldots, x_l) = \sum_{N \geqslant n_1 \geqslant \cdots \geqslant n_l \geqslant 1} \frac{x_1^{n_1}}{n_1^{k_1}} \cdots \frac{x_l^{n_l}}{n_l^{k_l}}. \tag{14}$$

Some indices are integer at $\varepsilon \to 0$ and some are half-integer; therefore, we get both $S(n; k; 1)$ and $S(2n; k; 1)$. We can re-express $S(n; k; 1)$ via $S(2n; k; \pm 1)$ by inserting $(1 + (-1)^{n_1})/2$ under the summation sign:

$${}_3F_2 \left(\begin{matrix} \frac{1}{2}, 1 + 2\varepsilon, -\varepsilon \\ \frac{3}{2} + 2\varepsilon, 1 - \varepsilon \end{matrix} \middle| 1 \right) = 1$$

$$- \varepsilon(1 + 4\varepsilon) \sum_{n=1}^{\infty} \frac{1}{(n-\varepsilon)(2n+1+4\varepsilon)} \exp\left[-\sum_{k=1}^{\infty} \frac{(-4\varepsilon)^k}{k} S(2n; k; -1) \right],$$

$${}_4F_3 \left(\begin{matrix} 1, \frac{1}{2} - \varepsilon, 1 + \varepsilon, -2\varepsilon \\ \frac{3}{2} + \varepsilon, 1 - \varepsilon, 1 - 2\varepsilon \end{matrix} \middle| 1 \right) = 1$$

$$- 2\varepsilon(1 + 2\varepsilon) \sum_{n=1}^{\infty} \frac{1}{(n-2\varepsilon)(2n+1+2\varepsilon)} \exp\left[\sum_{k=1}^{\infty} \frac{(2\varepsilon)^k}{k} \left(1 - (-1)^k\right) S(2n; k; -1) \right] \tag{15}$$

Products of S-sums with upper limit $2n$ are expressed in terms of single S-sums by means of the well-known algebra [9, 10, 11]. We expand the rational factors in Eq. (15) in ε, and then expand them into partial fractions. After that, separate sums may diverge, and we

introduce an upper limit N instead of ∞ in the outermost sum. Sums with $1/n^k$ can be re-written as sums in n to $2N$ by inserting $(1 + (-1)^n)/2$, and those with $1/(2n + 1)^k$ — by inserting $(1 - (-1)^n)/2$:

$$\sum_{n=1}^{N} \frac{1}{n^k} S(2n; k_1, \ldots, k_l; x_1, \ldots, x_l)$$

$$= 2^{k-1} [S(2N; k, k_1, \ldots, k_l; 1, x_1, \ldots, x_l) + S(2N; k, k_1, \ldots, k_l; -1, x_1, \ldots, x_l)],$$

$$\sum_{n=1}^{N} \frac{1}{(2n + 1)^k} S(2n; k_1, \ldots, k_l; x_1, \ldots, x_l)$$

$$= \frac{1}{2} [S(2N; k, k_1, \ldots, k_l; 1, x_1, \ldots, x_l) - S(2N; k, k_1, \ldots, k_l; -1, x_1, \ldots, x_l)$$

$$- S(2N; k + k_1, k_2, \ldots, k_l; x_1, x_2, \ldots, x_l) + S(2N; k + k_1, k_2, \ldots, k_l; -x_1, x_2, \ldots, x_l)]$$

(terms vanishing at $N \to \infty$ are omitted here). After that, all sums divergent at $N \to \infty$ cancel; in the remaining sums, we may set $N = \infty$. They are related to the Euler–Zagier sums [9, 10].

Yet another method of expanding certain classes of hypergeometric functions about half-integer parameters is provided by the algorithm of Ref. [8], by means of which we have calculated the expansion of Eq. (13) up to order $\mathcal{O}(\varepsilon^7)$. After factoring out an appropriate combination of pre-factors, the expansion reads

$$\frac{I_{122}}{\Gamma^3(1 + \varepsilon)} = \frac{\pi^2}{3(1 + 6\varepsilon)}$$

$$\times \left[1 - \pi^2 \varepsilon^2 + 48\zeta_3 \varepsilon^3 - \frac{38\pi^4}{15} \varepsilon^4 - 48 \left(\pi^2 \zeta_3 - 30\zeta_5 \right) \varepsilon^5 + \left(1152\zeta_3^2 - \frac{4793\pi^6}{945} \right) \varepsilon^6 \right.$$

$$\left. + \left(39312\zeta_7 - \frac{608\pi^4\zeta_3}{5} - 1440\pi^2\zeta_5 \right) \varepsilon^7 + \mathcal{O}(\varepsilon^8) \right]. \tag{16}$$

In order to check the correctness of this expansion, we have converted the hypergeometric functions in Eq. (13) to single Mellin-Barnes representations and subsequently obtained the coefficients numerically [12]. We find agreement to at least 14 decimal digits.

It is now straightforward to obtain the expansion of the master integral I_2 from Eq. (12):

$$\frac{I_2}{\Gamma^3(1 + \varepsilon)} = -\frac{\pi^2}{6} \left[\frac{1}{\varepsilon} + \frac{5}{2} - \left(\pi^2 + \frac{21}{4} \right) \varepsilon + \left(48\zeta_3 - \frac{5}{2}\pi^2 - \frac{599}{8} \right) \varepsilon^2 + \cdots \right]. \tag{17}$$

The first two terms have been obtained in Ref. [1] by a completely different method. The next one has been found by A. Pak from the requirement of cancellation of $1/\varepsilon$ poles in a physical calculation. The last term is then required to calculate the corresponding finite part. Yet higher terms in the expansion of I_2 can be obtained from Eq. (16).

As a closing remark we would like to mention that the expansion in Eq. (16) is found to agree up to order $\mathcal{O}(\varepsilon^7)$ with the expansion of the simple formula

$$\frac{I_{122}}{\Gamma^3(1 + \varepsilon)} = \frac{\pi^2}{3} \frac{\Gamma^3(1 + 2\varepsilon)\Gamma^2(1 + 3\varepsilon)}{\Gamma^6(1 + \varepsilon)\Gamma(2 + 6\varepsilon)}. \tag{18}$$

We do not know any analytical proof of this result, but high-precision numerical tests of Eqs. (13) and (18) for various values of ε on the real axis and in the complex plane strengthen the conjecture that this result might indeed be valid to all orders. However, until an analytical proof of Eq. (18) is found, we are guaranteed to obtain the correct expansion only by the analytically derived expression (13).

Acknowledgments

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